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Abstract

A new method that combines staggered grid Arbitrary Lagrangian-Eulerian (ALE) techniques with structured local adaptive mesh refinement (AMR) has been developed for solution of the Euler equations. The novel components of the combined ALE-AMR method hinge upon the integration of traditional AMR techniques with both staggered grid Lagrangian operators as well as elliptic relaxation operators on moving, deforming mesh hierarchies. Numerical examples demonstrate the utility of the method in performing detailed three-dimensional shock-driven instability calculations.

Keywords: arbitrary Lagrangian-Eulerian, ALE, adaptive mesh refinement, AMR, Lagrangian methods, CFD, inertial confinement fusion, ICF.

1 Introduction

The numerical simulation of compressible flows with shocks and material discontinuities is a computational challenge in many important application areas including inertial confinement fusion (ICF), astrophysics, and plasma physics. Lagrangian and ALE techniques have

often been favored in the above application areas [3], in part due to the self-adapting nature of Lagrangian grid motion, e.g., contact discontinuities are tracked automatically, and cells are clustered into high density regions behind shocks. However, this inherent form of adaption present in Lagrangian and ALE methods, is less general and robust than a dynamically adaptive method in which the number of cells may change with time, such as the structured grid local adaptive mesh refinement (AMR) methods[1, 5, 4, 11]. The development of a hybrid algorithm combining a Lagrange based ALE with AMR requires the development of modified methods for integration of the mesh hierarchy, new interlevel solution transfer operators, and methods for application of mesh relaxation operators to an AMR mesh hierarchy.

2 Equations of Motion and the Underlying ALE Method

The governing equations of inviscid gasdynamics are discretized from the Lagrangian form:

$$\frac{D\rho}{Dt} + \rho\vec{\nabla}\cdot\vec{V} = 0 \tag{1}$$

$$\rho\frac{D\vec{V}}{Dt} + \vec{\nabla}p = 0 \tag{2}$$

$$\rho\frac{De}{Dt} + p\vec{\nabla}\cdot\vec{V} = 0. \tag{3}$$

where ρ, e, p, and $\vec{V}$ are the fluid density, internal energy, pressure, and velocity respectively, and t is time.

The ALE method employed for integration of the system (1)-(2)-(3) is of the explicit, time-marching, Lagrange plus remap type. The initial Lagrange step follows the general approach taken by Tipton [14]. It employs a predictor-corrector discretization in time, and the HEMP spatial discretization [15, 13]. The scheme employs a monotonic artificial viscosity due to Christensen [9], and a kinematic hourglass filter [10]. The two-dimensional scheme has been described extensively previously; algorithmic details as well as comparisons with more widely known Eulerian methods can be found in a recent work

by Pember, et al.[12]. In the context of the adaptive method, a generalization to the definition of a nodal mass will be introduced.

At the end of a Lagrange step, it is often desirable to smooth the grid to prevent excessive mesh distortion which can lead to inaccuracy or even failure of the Lagrangian algorithm. An effective smoothing algorithm can be based upon a Laplace iteration for the transformed coordinates with respect to the Cartesian coordinates of each node. This is the essence of the Winslow method which we take as a representative relaxation operator. For the adaptive method, this must be applied to a mesh hierarchy, which introduces some additional considerations into the AMR hierarchy integration algorithm.

Once the relaxed mesh has been defined, it remains to interpolate the solution from the old Lagrange grid to the relaxed grid. We cast this interpolation in terms of an apparent advection equation. This advection equation is solved using a variant of the Corner Transport Upwind (CTU) scheme [2] for use on a staggered grid. The algorithmic details of the scheme have been discussed in detail in Pember [12], et al.

3 The Lagrangian (L-AMR) Algorithm

We develop first the adaptive components of the Lagrangian algorithm, and then extend the ideas to the ALE context. The essence of the adaptive Lagrangian method is the introduction of new interlevel solution transfer operators. Interlevel transfer operators are required when new grids are created, for the generation of pseudo boundary conditions on finer levels in the hierarchy, for synchronizing coarse and fine data in the hierarchy, and upon the removal of refined grids. The hierarchy advance for the Lagrangian algorithm requires no fundamental modification. However, care must be taken in applying coarse-fine boundary conditions on the moving mesh. On the fine mesh, the nodes coincident with the coarse mesh are slaved to the coarse node motion by interpolation in time, and the remaining "hanging nodes" in multi-dimensions are slaved by interpolating first in time, and then in space. The remaining ghost data are interpolated using the refinement operators to be described.

3.1 Staggered Mesh Refinement

The operators development here are designed with the following properties in mind:

P1) Constant field preservation

P2) 2nd order accuracy (in smooth regions)

P3) Monotonicity

P4) Local conservation

P5) Exact inversion of refinement by coarsening

A simple way to ensure that P4 and P5 are simultaneously achieved is to maintain an r:1, r being the refinement ratio, correspondence from fine nodes to coarse nodes, such that the local interpolation stencils on the fine mesh do not overlap. In this case inverting a locally conservative interpolation is simply a matter of summing the fine values of the conserved quantity in the stencil. This leads to a choice of odd refinement ratios only.

Consider a one-dimensional interpolation of some scalar density function ϕ with a known slope ϕ_0' and average value ϕ_0 over some interval Δx_0, into N arbitrary subintervals $\Delta x_k = x_{k+1} - x_k$.

An interpolation in which values are taken from the centers of the subintervals

$$\phi_k = \phi_0 + \phi_0' \left(\bar{x}_k - \frac{1}{2}\Delta x_0 \right)$$

where $\bar{x}_k = (x_k + x_{k+1})/2$, is locally conservative of $\phi\Delta x$ in the sense that

$$\sum_{k=1}^{N} \phi_k \Delta x_k = \phi_0 \Delta x_0,$$

since

$$\sum_{k=1}^{N} \phi_k \Delta x_k = \phi_0 \Delta x_0 + \phi_0' \left(\sum_{k=1}^{N} \bar{x}_k \Delta x_k - \frac{1}{2}\Delta x_0^2 \right). \tag{4}$$

In a constant field, all slopes ϕ_0' are zero, and constant fields are preserved independently of the mesh. We now have a general one-dimensional expression for interpolation that satisfies P1, P2, and P5. In order to satisfy P3, we employ the well-known van Leer limiter for slope determination.

If we desire to prevent oscillations in the primitive variables $\phi = (\rho, u, v, E)$, where E is the total energy, the required interpolation

basis to obtain property P4, local conservation, is $x = (V, \tilde{m}, \tilde{m}, m)$, where V is volume, $\tilde{m}$ is nodal mass, and m is cell mass. The multi-dimensional case is handled with d one-dimensional sweeps, or, if a strictly symmetric operator is desired, the average of $d!$ permutations of d sweeps.

Upon closer examination of (4), there is a consistency condition for local conservation that requires that the basis itself be locally conserved, i.e.,

$$\sum_{k=1}^{N} \Delta x_k = \Delta x_0. \tag{5}$$

We have identified two potential difficulties in achieving the consistency condition and hence a conservative operator. The first is calculation of hexahedral volumes in three dimensions. Many volume formulas for hexahedra are based on a surface triangulation of faces, which will not be consistent with a multi-linear interpolation of the mesh in the sense of (5). If instead one employs a bilinear surface model for the cell volume, then the interpolation is indeed consistent and mass conservation is retained.

However, if we employ a cell mass interpolation, (5) is violated in the case of a nodal mass basis for velocity interpolation. In this context the HEMP definition of nodal mass is overconstraining. If we instead generalize the definition of nodal mass by employing the concept of a "corner mass" as introduced in several other contexts [6, 7, 8], and apply our mass interpolation operator directly on the corner grid, we can retain both cell and nodal mass conservation. This introduction requires an analogous reformulation of the remap procedure, in which mass is remapped directly on the corner grid, and transport masses are then aggregated from the corner mesh, rather than averaged, for the velocity remap. The remap of total energy then requires a per-corner mass weighted definition of kinetic energy for defining a cell total energy, as opposed to simple averaging.

3.2 Coarsening

There are two natural choices for coarsening operators as weighted sums of the conserved quantities, i.e.,

$$\rho_0 = \frac{\sum \rho_i V_i}{\sum V_i} \text{ or } \frac{\sum \rho_i V_i}{V_0}$$

$$u_0 = \frac{\sum u_i \tilde{m}_i}{\sum \tilde{m}_i} \text{ or } \frac{\sum u_i \tilde{m}_i}{\tilde{m}_0}$$
$$E_0 = \frac{\sum E_i m_i}{\sum m_i} \text{ or } \frac{\sum E_i m_i}{m_0}$$

where i varies over the refinement stencil corresponding to each coarse node. The coarse mesh is formed by selection of every r'th mesh point. The first choice is a constant field preserving construction and the second choice is conservative, but not vice versa, in general. In order to achieve a simultaneously constant field preserving and conservative operator, one must apply a preprocessing remap operation to the fine grid data, remapping from the actual Lagrange grid to one which is fully aligned with the underlying coarse grid. The remapped data is discarded and does not replace the Lagrange solution on the fine level.

A demonstration of the three-dimensional L-AMR algorithm for the Taylor-Sedov blast wave is shown in Figure 1. This solution was computed using two mesh refinement levels. The coarsest level is shown in black and the fine level in white. The refined regions capture the outgoing shock as well as the region of strong expansion near the origin.

4 The Arbitrary Lagrangian-Eulerian (ALE-AMR) Algorithm

The introduction of a Winslow-type relaxation operator introduces some additional requirements for the ALE-AMR method. The relevant feature of the equipotential type methods is that they are derived from elliptic equations which intrinsically exhibit globally coupled solutions. The consequence for an AMR mesh hierarchy is that coarse meshes may not be relaxed independently of finer meshes; they instead require a solution method which enforces the required coupling between mesh levels. Thus one cannot simply use a composite Lagrange plus remap operator directly in the previously described L-AMR method to arrive at a well-behaved ALE-AMR method.

Instead the hierarchy advance algorithm is modified to include relaxation iterations *only when all finer levels have advanced to a given simulation time*. For a given level l_k, we wish the relaxation operator to behave as if it were applied to a "collapsed" hierarachy from l_k to the finest level l_N. This is accomplished with a mark-and-constrain strategy, in which the relaxation is applied to l_N, the resulting node motion

is injected to the next coarser level, and those coarse level nodes are marked. The unmarked nodes are then relaxed on the coarser level, and so on until l_k has been relaxed. This procedure comprises a single iteration of the relaxation operator. Additional iterations start again at the top of the hierarchy, such that mesh motion at all levels is coupled for each iteration.

5 Numerical Example

The utility of the algorithm is demonstrated on a three-dimensional Richtmeyer-Meshkov instability in a converging geometry; this serves as a proof-of-principle calculation for inertial confinement fusion (ICF) applications. The initial condition consists of an incoming spherical shock impinging upon a high density spherical shell with a single mode perturbation on its outer surface. For efficiency, the domain is constrained by cutting planes to approximately three wavelengths, as shown in Figure 2. As the shock impinges on the high density fluid, the classical bubbles and spikes instability occurs as shown in Figure 3. Later times show the growth of complex flow features, as shown in Figure 4. ALE-AMR enables much greater resolution on the detailed features of the instability than a comparable ALE method, and the mapped grid capability greatly reduces the "mesh imprinting" errors that tend to feed spurious anisotropic instability modes in this type of calculation on Cartesians grids.

6 Conclusion

The hybridization of staggered grid ALE and AMR on structured meshes is accomplished with the development of new interlevel transfer operators, the application of the idea of a "corner zone" to interpolation, modifications to the hierarchy advance algorithm, and methods for applying elliptic relaxation operators to a time- and space-refined mesh hierarchy. The advantageous features of both types of methods are retained, and the result is a powerful combination for the class of applications for which ALE methods are well-suited.

7 Acknowledgment

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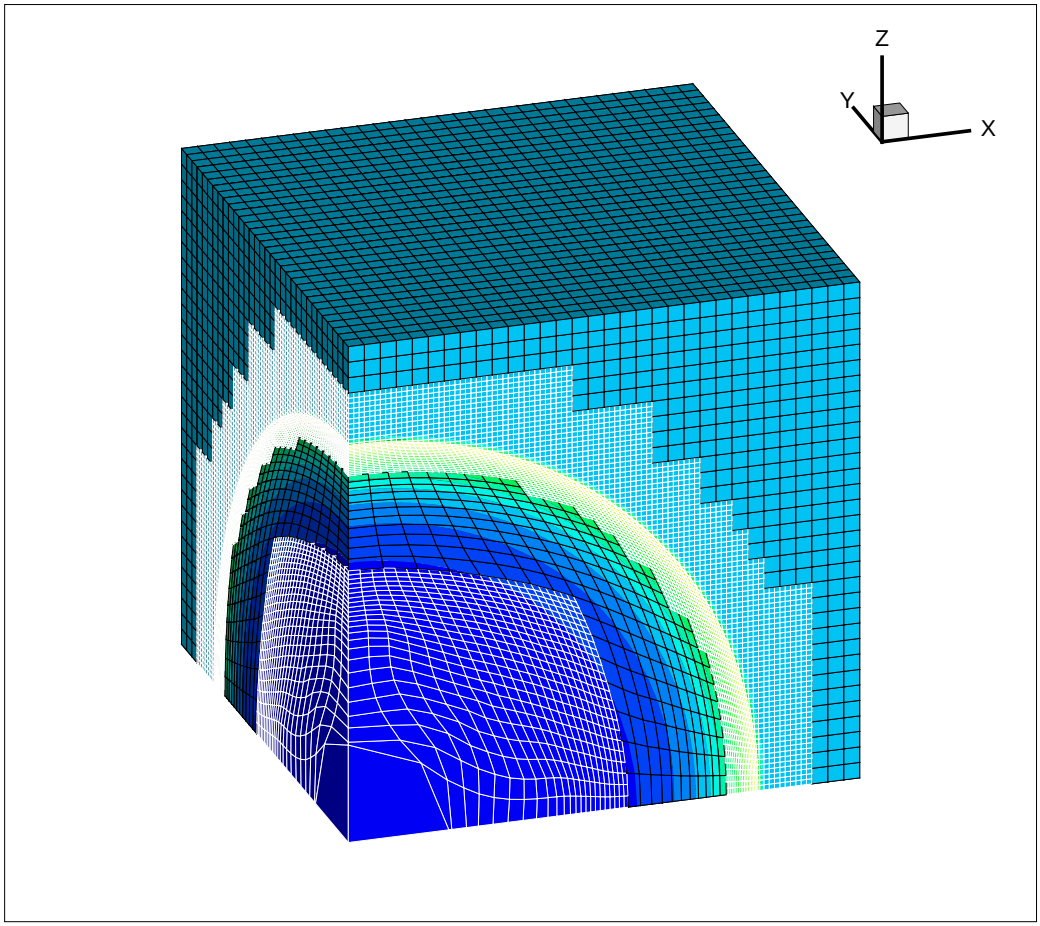

Figure 1: L-AMR solution of Taylor-Sedov blast wave problem. Coarse level grids shown in black, fine level grids in white. Colormap is density field.

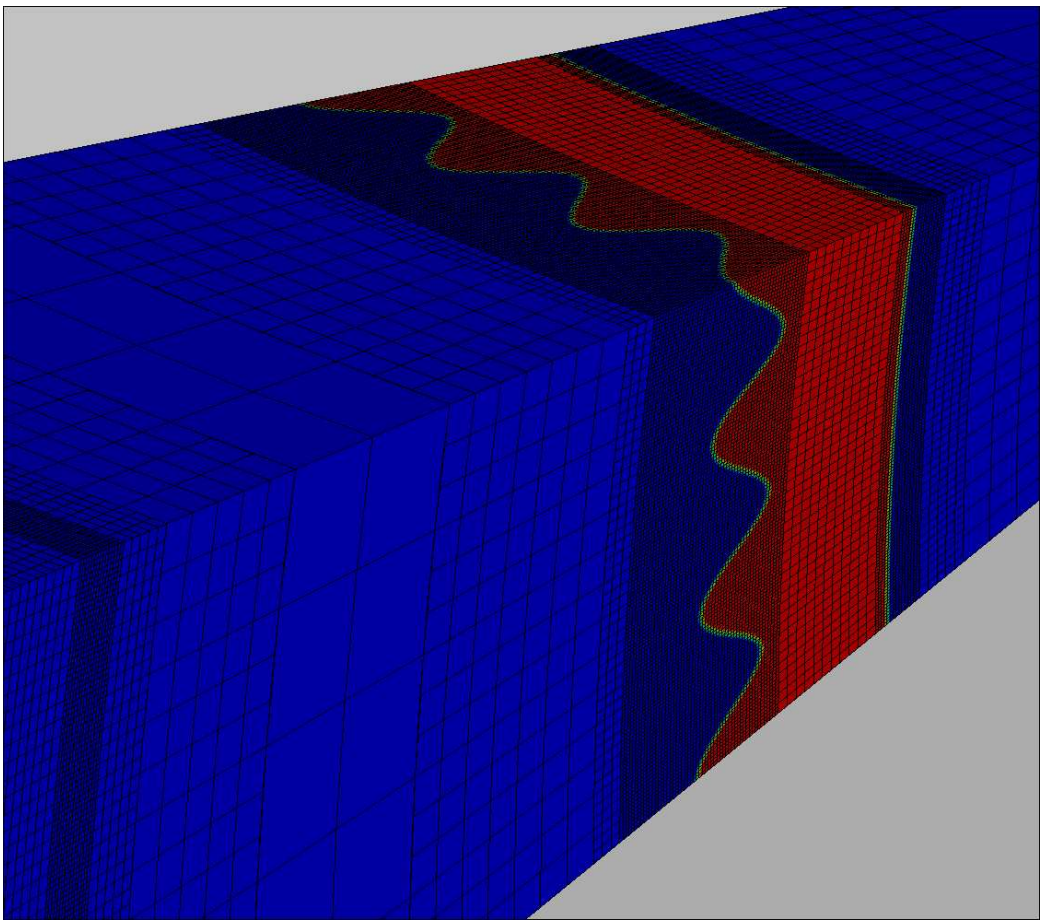

Figure 2: AMR grid hierarchy for initial condition of Richtmeyer-Meshkov instability calculation. Four levels of refinement. Colormap of density.

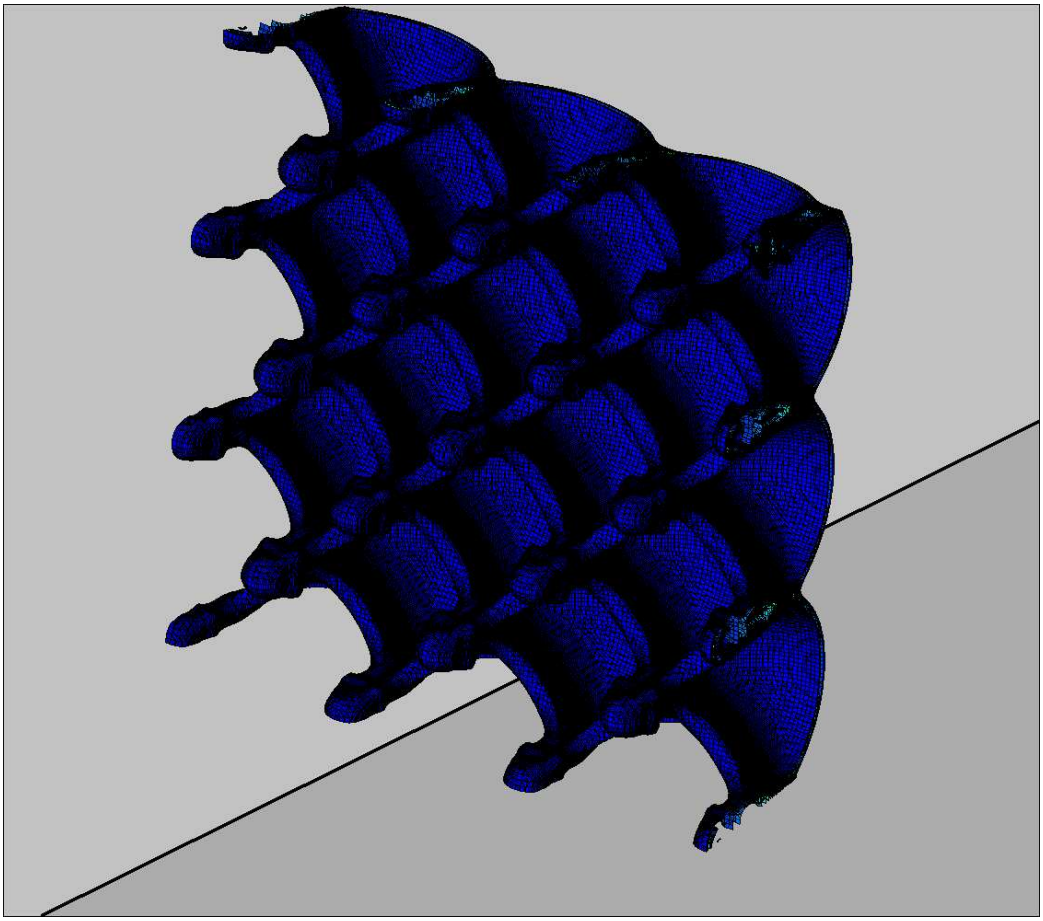

Figure 3: Bubbles and spikes instability. Isosurfaces of density.

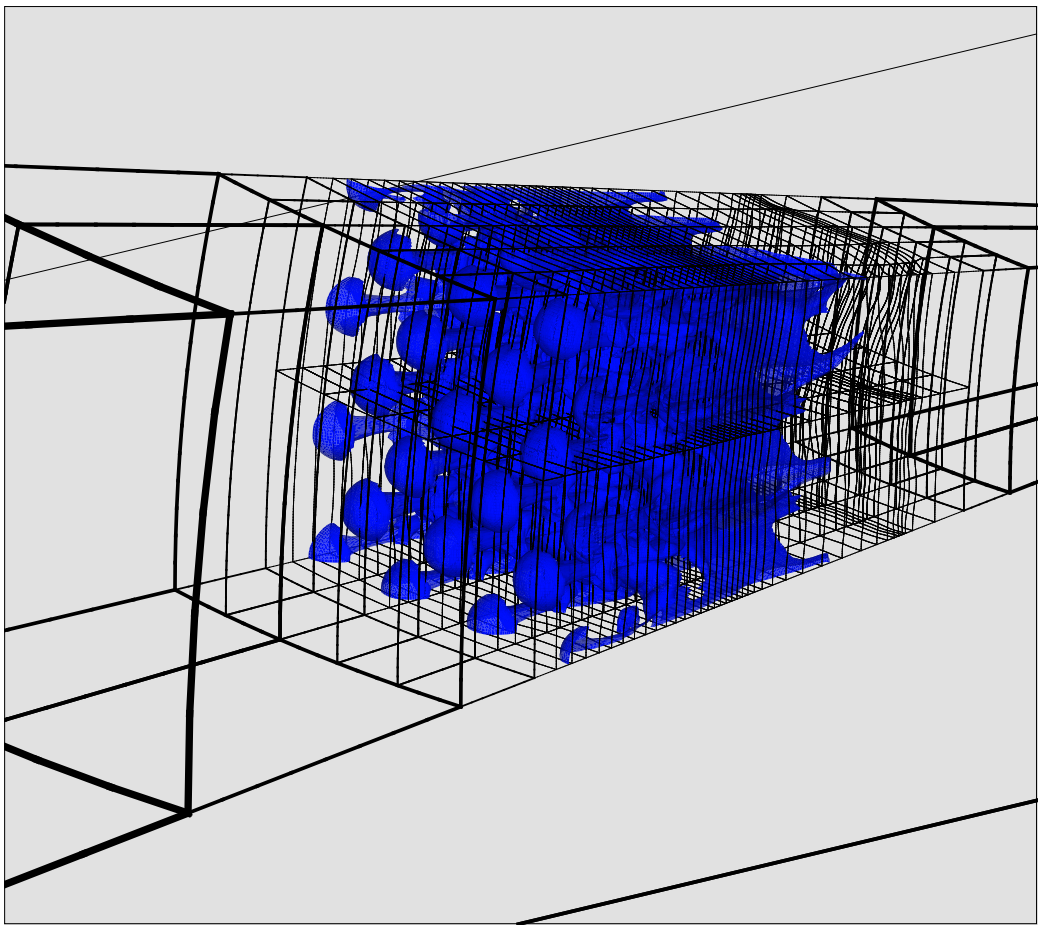

Figure 4: Later time instability growth. Isosurface of density. Box outlines shown in light to heavy outlines, from finest to coarsest level.